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On the construction of Strong Complementarity Slackness Solutions for DEA Linear Programming Problems using a Primal-Dual Interior-Point Method *

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Abstract

A novel approach for solving the DEA linear programming problems using a primal-dual interior-point method is presented. The solution found by this method satisfies the Strong Complementarity Slackness Condition (SCSC) and maximizes the product of the positive components among all SCSC solutions. The first property is critical in the use of DEA and the second one contributes significantly to the reliability of the solution.

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1 Introduction

Data Envelopment Analysis (DEA) is a linear programming application from the economic and management sciences where a desired solution of the underlying linear programming model is one satisfying the strong complementarity slackness condition (a SCSC solution). The set of SCSC solutions can be characterized as the set of solutions where the number of nonzero variables is maximal, therefore SCSC solutions are not vertex (or basic) solutions unless the solutions for both the primal and dual linear programming problems are unique. Thus, the standard simplex-type algorithms are generally not appropriate for computing SCSC solutions.

In this work we propose a primal-dual interior-point algorithm for solving the DEA linear programming problems. The solution found by this interior-point algorithm is not only a SCSC solution but it is also the center-most solution in the sense that it maximizes the product of the positive components among all the SCSC solutions (the so-called analytic center of the solution set). These properties are crucial in the use and reliability of DEA since among those SCSC solutions it is desirable to compute one that is in some sense as far away as possible from the relative boundary, avoiding in this way that small changes in the data may affect the DEA results (see Thompson, Dharmapala, and Thrall [16]).

This paper is organized as follows. In the next section we introduce DEA and give a formal definition of the SCSC solutions. Section 3 contains fundamental background for the proposed algorithm. In Section 4 we present the algorithm and we study its behavior in Section 5. Numerical results considering real world DEA problems are included in Section 6. Finally, in Section 7 we give some concluding remarks.

2 Data Envelopment Analysis (DEA)

DEA, as presented in Charnes and Cooper [2], is a technique for measuring the amount of relative efficiencies and inefficiencies in a production system characterized by data units called Decision Making Units (DMU).

Each Decision Making Unit DMU is a vector $\begin{pmatrix} y \\ -x \end{pmatrix} \in \mathbf{R}^{s+m}$ where $y \in \mathbf{R}^s, x \in \mathbf{R}^m, (y,x) \geq 0$ and neither y nor x is zero.

The amounts of output r and input i for DMU_j are represented respectively by y_{rj} and x_{ij}.

A DEA Data Domain D considers a set of n decision making units $DMU_1, \ldots, DMU_n$ and a collection of technologies which is characterized by a data matrix $P = (P_1 ... P_n) = \begin{pmatrix} Y \\ -X \end{pmatrix} \in \mathbf{R}^{(m+s)\times n}$ with $P_j = DMU_j$, $Y = (y_1 ... y_n) \in \mathbf{R}^{s\times n}$ and $X = (x_1 ... x_n) \in \mathbf{R}^{m\times n}$.

A standard assumption for DEA is that no two columns of P are proportional.

The production possibility set associated to the data matrix P is defined as

$$\mathcal{K}(\mathcal{P}) = \left\{ p = \begin{pmatrix} y \\ -x \end{pmatrix} \in \mathbf{R}^{s+m} : p \leq P\lambda \text{ for some } \lambda \geq 0, \lambda \in \mathbf{R}^n, \text{ and } (y,x) \geq 0 \right\}.$$

Let $DMU_o = \begin{pmatrix} y_o \\ -x_o \end{pmatrix}$ denote one of the $DMU's$ (i.e. $o \in \{1, \ldots, n\}$).

Consider the linear programming problem associated to DMU_o

$$\begin{aligned} &\text{minimize} && \theta \\ &\text{subject to} && S = P\lambda - \begin{pmatrix} y_o \\ -\theta x_o \end{pmatrix} \geq 0, \\ & && \lambda \geq 0, \end{aligned} \tag{1}$$

where $\theta \in \mathbf{R}$, $\lambda \in \mathbf{R}^n$ and $S \in \mathbf{R}^{s+m}$.

Problem (1) is referred to as the envelopment (or primal) program for DMU_o.

This problem is the CCR ratio model introduced by Charnes, Cooper, and Rhodes (1978).

The present paper focuses attention on this DEA model although our analysis can be extended to the BCC linear model (Banker, Charnes, and Cooper (1984)).

Let $(\theta_o^*, \lambda_o^*, S_o^*)$ be a solution of (1). The DEA-radial-efficiency for DMU_o is defined as the optimal value θ_o^*.

The DMU_o is said to be DEA-radial-efficient if $\theta_o^* = 1$ and the DMU_o is said to be DEA-radial-inefficient if $\theta_o^* < 1$. This definition is equivalent to saying that a DMU is DEA-radial-efficient if it lies in the frontier of the production possibility set.

The definition of efficiency used in DEA is based on the idea that to remain in the production possibility set a DMU is technical efficient if the inputs and outputs, corresponding to that DMU, can not be respectively decreased nor increased. Therefore, a DMU_o can be a boundary point ($\theta_o^* = 1$) and not be efficient. For this reason, a given DMU_o is called DEA-efficient if $\theta_o^* = 1$ and $S_o^* = 0$ for all the solutions $(\theta_o^*, S_o^*, \lambda_o^*)$ of problem (1).

There is an alternative dual approach to DEA-radial-efficiency for DMU_o using the multiplier space $\mathcal{W}_o$ defined as

$$\mathcal{W}_o = \{w = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbf{R}^{s+m} : h_o(w) \geq h_j(w) \text{ for all } j = 1, \ldots, n, w \geq 0 \text{ and } u \neq 0, v \neq 0\}.$$

Here the functions h_j are output/input ratios defined as

$$h_j(w) = u^T y_j / v^T x_j \text{ for } j = 1, \ldots, n,$$

for each $w = \begin{pmatrix} u \\ v \end{pmatrix}, u \in \mathbf{R}^s, v \in \mathbf{R}^m, w \geq 0$ such that $v^T x_j > 0$.

Let

$$\mathcal{W}_o^m = \{w \in \mathcal{W}_o : v^T x_o = u^T y_o = 1\}.$$

The set $\mathcal{W}_o^m$ is called the normalized multiplier set for DMU_o.

Observe that $\mathcal{W}_o^m \neq \emptyset$ if and only if $\mathcal{W}_o \neq \emptyset$.

Consider the linear programming problem

$$
\begin{array}{ll}
\text{maximize} & u^T y_o \\
\text{subject to} & v^T x_o = 1, \\
& -t_j = u^T y_j - v^T x_j = w^T P_j \leq 0, \quad j = 1, \ldots, n \\
& w = \begin{pmatrix} u \\ v \end{pmatrix} \geq 0,
\end{array} \tag{2}
$$

where $u \in \mathbf{R}^s$, $v \in \mathbf{R}^m$ and $t \in \mathbf{R}^n$.

Problem (2) is the dual problem of (1) and it is referred to as the multiplier (or dual) program for DMU_o.

It is easy to see that (w_o^*, t_o^*) is a solution of problem (2) if and only if $w^* \in \mathcal{W}_o^m$. Therefore, by duality theory, $\mathcal{W}_o \neq \emptyset$ if and only if $\theta_o^* = 1$. Hence, DMU_o is DEA-radial-efficient if $\mathcal{W}_o \neq \emptyset$ and is DEA-radial- inefficient if $\mathcal{W}_o = \emptyset$.

Let $\mathcal{RE}$ be the set of all the DEA-radial-efficient DMU_j and $\mathcal{N}$ be the set of all the DEA-radial-inefficients DMU_j, for $j = 1, \ldots, n$.

The DEA-radial-efficiency can be computed using either of the two linear programs (1) and (2). Hence, the solutions of these problems answer the question of DEA-radial-efficiency. However, among those $DMU's$ that are DEA-radial-efficient (or DEA-radial-inefficient), there are important differences depending on their multiplier sets. Thus, the set of $DMU's$ may be partitioned into six classes (see [3] and [4] for more detail):

$$E = \{DMU_j \in \mathcal{RE} : \text{ dim } \mathcal{W}_j = s + m\},$$

$$E' = \{DMU_j \in \mathcal{RE} : \text{ dim } \mathcal{W}_j < s + m \text{ and exist } w > 0, w \in \mathcal{W}_j\},$$

$$F = \{DMU_j \in \mathcal{RE} : \text{ every } w \in \mathcal{W}_j \text{ has at least one zero component}\},$$

$$NE = \{DMU_j \in \mathcal{N} : DMU_j' \in E\},$$

$$NE' = \{DMU_j \in \mathcal{N} : DMU_j' \in E'\},$$

$$NF = \{DMU_j \in \mathcal{N} : DMU_j' \in F\}.$$

Here $DMU'_j = P_j(\theta^*_j) = \begin{pmatrix} y_j \\ -\theta^*_j x_j \end{pmatrix}$.

The elements of E, E', and F are called, respectively, DEA-extreme efficient, DEA-non-extreme efficient, and DEA-weak efficient. It can be seen [4] that any given DMU_o is DEA-efficient if it belongs to $E \cup E'$, or equivalently, if there exists a positive multiplier vector $w \in \mathcal{W}_o$.

The classification and characterization of the $DMU's$ play a very important role in DEA. Charnes, Cooper, and Thrall [4] present a structure for this characterization using the linear programming problems (1) and (2). A very interesting result from [4] is Lemma 10A which gives a way to classify the different DMU's, but requires the computation of a solution $(\theta_o, \lambda_o, S_o, w_o, t_o)$ of problems (1) and (2) satisfying

$$S_o^T w_o = \lambda_o^T t_o = 0$$

and

$$S_o + w_o > 0, t_o + \lambda_o > 0.$$

These solutions are said to satisfy the strong complementarity slackness condition (or are SCSC solutions). It is well known (see Spivey and Thrall[14] for a proof) that there always exists a SCSC solution for any linear programming problem which is both primal and dual feasible.

The power of the SCSC approach depends on how far w_o is from the boundary of the set $\mathcal{W}_o$. In order to explain this idea (see Thompson, Dharmapala, and Thrall [16] for more details) let us consider a DMU_o being DEA-extreme efficient (i.e., $DMU_o \in E$). Then, for every SCSC solution $(\theta_o, \lambda_o, S_o, w_o, t_o)$, w_o is the center of an $(m + s)$-dimensional sphere (hypersphere) which is a subset of $\mathcal{W}_o$ and for every point w'_o in this sphere, $h_o(w'_o) > h_j(w'_o)$ for all $j \neq o$. Therefore, one has the reasonable expectation that DMU_o remain DEA-extreme efficient for relative small changes in the data. Moreover, if we define the sensitivity

function

$$d_o(w_o) = h_o(w_o) - h_k(w_o), \text{ with } h_k(w_o) = \max_{j \neq 0} h_j(w_o)$$

we expect the DEA results to be less sensitive to changes in the data when $d_o(w_o)$ is larger. Therefore, for DEA purposes, the best choice among all the SCSC solutions is the one that maximizes the function $d_o(w)$ over the set of dual multipliers w with $(\theta, \lambda, S, w, t)$ a SCSC solution. Let us call this solution the DEA-center solution $(\theta_o^c, \lambda_o^c, S_o^c, w_o^c, t_o^c)$ and $d_o^c = d_o(w_o^c)$.

The DEA-center is an useful theoretical definition for understanding the idea of centrality that best fits the DEA requirements, however its practical use depends on the development of an effective numerical method for computing it. In this paper we propose an algorithm that computes a SCSC solution called the analytic center. This solution satisfies a notion of centrality that we hope is close to the idea of centrality given by the DEA-center, however the exact relationship between the analytic center and the DEA-center, as well as the computation of the DEA-center solution, are important topics that will not be covered by this paper and deserve further study.

3 Algorithm Background

Simplex-type methods consider the vertices of the feasibility set of the linear programming problem and the final solution is an optimal vertex. On the other hand, interior-point methods generate an infinite sequence of points in the interior of the feasibility set and the algorithm stops when the iterates are sufficiently close to a solution.

There are several different classes of interior-point methods for linear programming (see Gonzaga[7] for a survey). In this work we are interested in primal-dual interior-point algorithms. We introduce in this section some theoretical background concerning these algorithms as well as a discussion of the main results necessary to motivate the development of our algorithm.

We employ the standard notation used in the literature concerning primal-dual interior-point algorithms.

Consider the linear programming problem in the standard form:

$$\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{subject to} \quad Ax = b, \quad x \geq 0, \end{aligned} \tag{3}$$

where $c, x \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, $A \in \mathbf{R}^{m \times n} (m < n)$ and A has full rank m. Associated with problem (3) is the dual linear program

$$\begin{aligned} &\text{maximize} \quad b^T y \\ &\text{subject to} \quad A^T y + z = c, \quad z \geq 0, \end{aligned} \tag{4}$$

where $y \in \mathbf{R}^m$ and $z \in \mathbf{R}^n$.

Let

$$\mathcal{F} = \{(x, y, z) : Ax = b, A^T y + z = c, (x, z) \geq 0\}.$$

A point $(x, y, z) \in \mathcal{F}$ is said to be strictly feasible if x and z are strictly positive.

It is well known that x is a solution for (3) and (y, z) is a solution for (4) if and only if $(x, y, z) \in \mathcal{F}$ and the duality gap $c^T x - b^T y = 0$, or equivalently, $x^T z = 0$. This latter condition is called the complementarity slackness condition.

Primal-dual interior-point algorithms attempt to solve the primal and dual linear programs simultaneously by generating a sequence of strictly feasible points (x_k, y_k, z_k) such that the duality gap $x_k^T z_k$ converges to zero.

Observe that $x^T z = 0$ with $(x, z) \geq 0$ if and only if $x_i z_i = 0$ for all $i = 1, \ldots, n$. The optimality conditions for (3) and (4) can be written as

$$F(x, y, z) = \begin{pmatrix} Ax - b \\ A^T y + z - c \\ XZe \end{pmatrix} = 0, \quad (x, z) \geq 0, \tag{5}$$

where $X = diag(x)$, $Z = diag(z)$ and $e = (1, 1, \ldots, 1, 1)^T \in \mathbf{R}^n$.

Most primal-dual interior-point algorithms are variants of Newton's method applied to problem (5).

We denote the solution set of problem (5) by

$$\mathcal{S} = \{(x, y, z) : F(x, y, z) = 0, (x, z) \geq 0\}.$$

A point (x, y, z) in $\mathcal{S}$ is said to be a strict complementarity solution if $x_i + z_i > 0$ for all $i = 1, \ldots, n$.

For a given DMU_o, the DEA problems (1) and (2) discussed in the previous section can be written in the standard forms (3) and (4) by defining

$$A = \begin{pmatrix} -I_{m\times m} & 0 & 0 & P \\ 0 & -I_{s\times s} & x_o & \end{pmatrix} \in \mathbf{R}^{(m+s)\times(m+n+s+1)}, c = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbf{R}^{m+n+s+1}, b = \begin{pmatrix} y_o \\ 0 \end{pmatrix} \in \mathbf{R}^{m+s}.$$

Therefore, $x = \begin{pmatrix} S \\ \theta \\ \lambda \end{pmatrix} \in \mathbf{R}^{m+n+s+1}$, and $z = \begin{pmatrix} w \\ 0 \\ t \end{pmatrix} \in \mathbf{R}^{m+n+s+1}$.

Thus, the set of strict complementarity solutions is equal to the set of SCSC solutions.

In this work we assume the following standard conditions for (3) and (4):

(A1) $\{x \in \mathbf{R}^n : Ax = b, x > 0\} \neq \emptyset$, and

(A2) $\left\{(y, z) \in \mathbf{R}^{m+n} : A^T y + z = c, z > 0\right\} \neq \emptyset$.

We are particularly concerned with the case when $\mathcal{S}$ is not a singleton set, i.e.,

$$ri(\mathcal{S}) \neq \emptyset$$

where $ri(\mathcal{S})$ denotes the relative interior of $\mathcal{S}$. In this case, both problems (3) and (4) do not have unique solutions.

Under the previous assumptions, the solution set S has the following interesting structure: (i) $\mathcal{S} \neq \emptyset$ is bounded ; (ii) all points in the relative interior are SCSC solutions (and all points

on the relative boundary are not); (iii) the zero-nonzero pattern of points in the relative interior is invariant. See Zhang and Tapia [19], Charnes et al. [4] and El-Bakry et al. [5] for proofs.

Therefore, for any $(x^*, y^*, z^*) \in ri(\mathcal{S})$, the following index sets

$$I_x^+ = \{i : x_i^* > 0, 1 \leq i \leq n\} \text{ and } I_z^+ = \{i : z_i^* > 0, 1 \leq i \leq n\}$$

are independent of the choice of (x^*, y^*, z^*). Moreover, by strict complementarity

$$I_x^+ \bigcup I_z^+ = \{1, 2, ..., n\} \text{ and } I_x^+ \bigcap I_z^+ = \emptyset.$$

Because of this structure of the solution set $\mathcal{S}$ the solutions in $ri(\mathcal{S})$ may be characterized as those for which the number of nonzero components is maximal. Among them, there is a solution that may be thought of as the center solution in the sense that it maximizes the product of the positive components. This solution is called the analytic center of the solution set and it was introduced by McLinden [11] in a general setting and independently by Sonnevend [13] in the context of linear programming. It is interesting to mention that Goldman and Tucker [6] introduced a concept of central solution for game theory which may be related with the concept of analytic center for linear programming. However, the idea of centrality defined by the analytic center is not exactly the same as the central solution of a game as defined in [6].

Formally, the analytic center of the solution set $\mathcal{S}$ is defined as:

$$(x^*, y^*, z^*) = \arg\max \{\psi(x, z) : (x, y, z) \in \mathcal{S}\} \tag{6}$$

where

$$\psi(x, z) = \prod_{i \in I_x^+} x_i \prod_{i \in I_z^+} z_i.$$

Equivalently, in (6) one can replace $\psi(x, z)$ by its logarithm, i.e.,

$$\ln \psi(x, z) = \sum_{i \in I_x^+} \ln x_i + \sum_{i \in I_z^+} \ln z_i.$$

The central path of problem (5) is defined as the set

$$\mathcal{P}_c = \{(x(\mu), y(\mu), z(\mu)) \text{ is strictly feasible } : X(\mu)Z(\mu)e = \mu e\}, \text{ for } \mu > 0.$$

This is equivalent to saying that a strictly feasible point (x, y, z) is on the central path if and only if it satisfies $x_1 z_1 = x_2 z_2 = ... = x_n z_n$.

These notions of analytic center and central path are well-defined under the assumptions (A1), (A2). For more details see McLinden [11], Megiddo [12] and Gonzaga [7].

A very interesting result from McLinden [11] is Theorem 9. In the case of linear programming, it states that the central path intersects the solution set at the analytic center, i.e., the central path point $(x(\mu), y(\mu), z(\mu))$ converges to the analytic center (x^*, y^*, z^*) as μ converges to zero. See also Proposition 8.2 in Megiddo [12] and the discussion preceding it. This result has played an important role in the development of most primal-dual interior-point algorithms which attempt to follow the central path.

The interior-point algorithm proposed in this work has been designed for effectively computing the analytic center of the solution set in linear programming.

4 Algorithm

The first primal-dual interior-point method for linear programming was constructed by Kojima, Mizuno and Yoshise [9], based on the work of Megiddo [12]. The following general framework captures the essence of this algorithm and contains the majority of the primal-dual interior-point methods that can be found in the literature.

Algorithm 4.1 (Generic Kojima-Mizuno-Yoshise primal-dual algorithm)
Given a strictly feasible point (x^0, y^0, z^0). *For* $k = 0, 1, 2, \ldots,$ *do*

(1) *Choose* $\sigma^k \in [0, 1)$ *and set* $\mu^k = \sigma^k (x^k)^T z^k / n$.

(2) *Solve the following system for* $(\Delta x^k, \Delta y^k, \Delta z^k)$*:*

$$F'(x^k, y^k, z^k) \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = -F(x^k, y^k, z^k) + \mu^k \begin{pmatrix} 0 \\ 0 \\ e \end{pmatrix}.$$

(3) *Choose* $\tau^k \in (0,1)$ *and compute the step-length* $\alpha^k = \min(1, \tau^k \hat{\alpha}^k)$*, where*

$$\hat{\alpha}^k = \frac{-1}{\min((X^k)^{-1}\Delta x^k, (Z^k)^{-1}\Delta z^k)}.$$

(4) *Form the new iterate*

$$(x^{k+1}, y^{k+1}, z^{k+1}) = (x^k, y^k, z^k) + \alpha^k(\Delta x^k, \Delta y^k, \Delta z^k).$$

The algorithm proposed in this work is a modification of the Kojima, Mizuno and Yoshise algorithm, and is similar to one presented in Gonzaga [7].

The following lemma was proved in Zhang and Tapia [19]. It provides a sufficient condition for a strictly feasible sequence $\{(x^k, y^k, z^k)\}$ to converge to the analytic center of the solution set.

Lemma 4.1 (Zhang-Tapia) Let $\{(x^k, y^k, z^k)\}$ be a strictly feasible sequence and $e = (1, \ldots, 1)^T \in \mathbf{R}^n$. Assume

$$\mu^k = (x^k)^T z^k / n \to 0$$

and

$$\|X^k z^k / \mu^k - e\| \to 0. \tag{7}$$

Then $\{(x^k, y^k, z^k)\}$ converges to the analytic center of the solution set.

Our approach to constructing an algorithm for computing the analytic center is to try to enforce condition (7).

The proposed algorithm will utilize two key ingredients:

- The notion of β-neighborhoods of the central path, defined for some $\beta \in (0, 1/4]$ by

$$\mathcal{N}(\beta) = \{(x, y, z) : (x, y, z) \in \mathcal{F}, \ \|Xz - \mu e\| \leq \beta\mu\}, \tag{8}$$

where

$$\mu = x^T z/n.$$

- The use of a damped Newton method for solving the system

$$\mathcal{F}_\mu(x, y, z) = \begin{pmatrix} Ax - b \\ A^T y + z - c \\ XZe - \mu \end{pmatrix} = 0,$$

with $\mu > 0$, in order to get to the central path. A line-search technique is used as a globalization of Newton's method. For each $\mu > 0$, the merit function considered is defined by the function

$$f_\mu : \mathbf{R}^{n+m} \to \mathbf{R}$$

with

$$f_\mu(x, y, z) = \|\mathcal{F}_\mu(x, y, z)/\mu\|_2^2.$$

Observe that

$$f_\mu(x, y, z) = \|(Xz - \mu e)/\mu\|_2^2, \text{ for } (x, y, z) \in \mathcal{F}.$$

The function f_μ measures closeness to the central path.

The idea of the algorithm is to consider a subsequence of gradually shrinking neighborhoods $\{\mathcal{N}(\beta^{k_j})\}$ of the central path with $\beta^{k_j} \to 0$. The algorithm generates an iteration sequence $\{(x^k, y^k, z^k)\}$ such that a subsequence $(x^{k_j}, y^{k_j}, z^{k_j})$ belongs to $\mathcal{N}(\beta^{k_j})$. This subsequence is obtained by considering a sequence of $\mu's$ and for each fixed μ using a damped Newton method (with a line search globalization strategy) to solve the system $\mathcal{F}_\mu(x, y, z) = 0$.

A description of the algorithm follows:

Algorithm 4.2 (Long-Step Shrinking-Neighborhood Algorithm)

Given a strictly feasible point $w^0 = (x^0, y^0, z^0)$, $\sigma^0 \in (0,1)$ *and* $\beta^0 \in (0,1)$.

Choose $\eta \in (0, 1/2)$.

Do until convergence

(0) *Set* $k = 0$.

(1) *Define parameters: Set* $\mu = \sigma^0 (x^k)^T z^k / n$ *and* $\beta = \beta^k$.

(2) *Closeness to the central path: If*

$$\|X^k Z^k e/\mu - e\|_2 \leq \beta \tag{9}$$

go to **(6)**.

(3) *Compute new interate:*

(3.1) Solve the following system for $(\Delta x^k, \Delta y^k, \Delta z^k)$*:*

$$F'(x^k, y^k, z^k) \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = -F(x^k, y^k, z^k) + \mu \begin{pmatrix} 0 \\ 0 \\ e \end{pmatrix}.$$

(3.2) Choose $\tau^k \in (0,1)$ *and compute the step-length* $\alpha^k = \min(1, \tau^k \hat{\alpha}^k)$*, where*

$$\hat{\alpha}^k = \frac{-1}{\min((X^k)^{-1}\Delta x^k, (Z^k)^{-1}\Delta z^k)}.$$

(3.3) Form the iterate

$$w^{k+1} = (x^{k+1}, y^{k+1}, z^{k+1}) = (x^k, y^k, z^k) + \alpha^k (\Delta x^k, \Delta y^k, \Delta z^k).$$

(4) *Line search:*

(4.1) If $f_\mu(w^{k+1}) \leq f_\mu(w^k) - (\eta \alpha^k / 2\mu^2) \mathcal{F}_\mu^T(w^k) \mathcal{F}_\mu(w^k)$ *go to* **(5)**.

(4.2) If not, reduce α^k *and form the iterate*

$$w^{k+1} = (x^{k+1}, y^{k+1}, z^{k+1}) = (x^k, y^k, z^k) + \alpha^k(\Delta x^k, \Delta y^k, \Delta z^k).$$

(4.3) Go again to step (4.1).

(5) *Set* $k = k + 1$, $\beta^k = \beta$ *and go to* **(2)**.

(6) *Set* $\mu = \sigma^0 (x^k)^T z^k / n$.

(7) *Do (3.1), (3.2), (3.3).*

(8) *Decrease neighborhood: Choose* $\beta^{k+1} \leq \beta^k$.

(9) *Set* $k = k + 1$ *and go to* **(1)**.

5 Algorithm Behavior

In most primal-dual interior-point methods the search directions used are linear combinations of two fundamental directions, the so-called affine scaling or pure Newton direction (direction in which the gap is decreased) and the centering direction (direction towards the central path). This is also the case in our algorithm.

In this section we will describe how the proposed algorithm behaves with respect to the two objectives of decreasing the gap and centering the iterates.

The following propositions address these issues. Their proofs are straightforward (see Gonzalez-Lima and Tapia [8] for details).

Proposition 5.1 *Let* $\{(x^k, y^k, z^k)\}$ *and* $\{\alpha^k\}$ *be generated by the Long-Step Shrinking-Neighborhood Algorithm (Algorithm 4.2) and consider the sets*

$$I_{cp} = \{k \geq 0 : (x^k, y^k, z^k) \text{ satisfies condition (9) in the algorithm } \}, i.e.$$

$$I_{cp} = \{k \geq 0 : \|X^k Z^k e/\mu - e\|_2 \leq \beta^k\}, \textit{and}$$

$$I_{cp}^+ = \{k \geq 1 : (x^{k-1}, y^{k-1}, z^{k-1}) \textit{ satisfies condition (9) in the algorithm } \} \cup \{0\}.$$

For each k we denote by k_{cp}^+ the largest element in I_{cp}^+ such that $k_{cp}^+ \leq k$ and by k_{cp} the smallest element in I_{cp} such that $k \leq k_{cp}$.

Define

$$\sigma^k = \sigma^o / \prod_{m=k_{cp}^+}^{k-1} (1 - \alpha^m(1 - \sigma^m)), \textit{ for } k \notin I_{cp} \cup I_{cp}^+$$

and

$$\sigma^k = \sigma^o, \textit{ for } k \in I_{cp} \cup I_{cp}^+.$$

Then

(i) $\sigma^k \in (0, 1]$, *for all* $k = 0, 1, ..$

(ii) $\sigma^k = 1$ *if* $\alpha^{k-1} = 1$, *for* $k \notin I_{cp} \cup I_{cp}^+$

. $\sigma^{k_{cp}^+ + 1} \leq \sigma^{k_{cp}^+ + 2} \leq ... \leq \sigma^{k_{cp}} \leq 1.$

Proposition 5.2 *Let $\{(x^k, y^k, z^k)\}$ and $\{\alpha^k\}$ be generated by the Long-Step Shrinking-Neighborhood Algorithm (Algorithm 4.2) and let σ^k for all $k \geq 0$ be defined as in Proposition 5.1.*

Then

(i) *For all* $k = 0, 1, ...,$ $\mu = \sigma^k (x^k)^T z^k / n$

(ii) $(x^{k+1})^T z^{k+1} = (1 - \alpha^k(1 - \sigma^k))(x^k)^T z^k.$

Therefore,

$$(x^{k+1})^T z^{k+1} \leq (x^k)^T z^k, \textit{ for all } k = 0, 1, 2, ...,$$

If $\sigma^k = 1$, then $(x^{k+1})^T z^{k+1} = (x^k)^T z^k$ (the gap does not decrease). If $\sigma^k = 0$, then $(x^{k+1})^T z^{k+1} < (x^k)^T z^k$ (maximum possible gap decrease in that iteration).

Proposition 5.3 *Let $s^k = (\Delta x^k, \Delta y^k, \Delta z^k)$ be the step direction computed by the Long-Step Shrinking-Neighborhood Algorithm (Algorithm 4.2), with $\mu > 0$ fixed, and*

$$f_\mu(x, y, z) = \|(Xz - \mu e)/\mu\|_2^2.$$

Then

s^k is a descent direction for $f_\mu(x, y, z)$ at (x^k, y^k, z^k).

The previous propositions state that while condition (9) (closeness to the central path) in Algorithm 4.2 is not satisfied, μ will be left fixed and the dominant direction will be the centering direction in the sense that the amount of gap decrease will be smaller at each iteration (as a consequence of the increasing values of the centering parameters $\sigma's$).

Once condition (9) is satisfied, the value of μ will be decreased and the gap will decrease by a factor close to or equal to the value of σ^o, in two contiguous iterations.

The process will continue in this way with the interesting feature that the satisfaction of condition (9) will force the iterates to stay close to the central path when the gap is small. Then, condition (7) from Lemma 4.1 will be satisfied for a subsequence of the iteration sequence $\{(x^k, y^k, z^k)\}$ generated by the algorithm.

6 Numerical Experience

In this section we discuss the numerical results obtained by using the Long-Step Shrinking-Neighborhood Algorithm for DEA problems.

The experiments were performed in 64 bit arithmetic using a code implemented in MATLAB[1]. The code is implemented following Lustig, Marsden and Shanno [10], so the starting point is not necessarily feasible. The code generates a sequence of iterates that approach feasibility and drive the gap to zero.

[1]Trademark of The Math Works, Inc., 21 Eliot Street South Natick, MA 01760.

We say that a problem is solved to an accuracy of 10^{-m} for some positive integer m if the algorithm is terminated when

$$\max\left(\frac{|c^t x^k - b^T y^k|}{1+|b^T y^k|}, \frac{\|Ax^k - b\|_1}{1+\|x\|_1}, \frac{\|A^T y^k + z^k - c\|_1}{1+\|y^k\|_1 + \|z^k\|_1}, \frac{\|X^k z^k - e\|_2}{x^T z/n}\right) \leq 10^{-m}.$$

In this study all the problems were solved to an accuracy of 10^{-8} using a Sun 4/490 workstation. The algorithm stops when the problem is solved to the given accuracy or when the number of iterations is equal to 200.

6.1 Classification of the *DMU's*

There are two main objectives in this section. The first one is to show that the solution computed by using the Long-Step Shrinking-Neighborhood Algorithm is the analytic center solution for each of the considered problems. The second objective is to use the computed solution and the following Lemma (Lemma 10A from [4]) in order to classify the *DMU's* of each problem into the six classes E, E', F, NE, NE', NF.

Lemma 6.2 (Charnes, Cooper, Thrall) *Let $(\theta_o, \lambda_o, S_o)$ and (u_o, v_o, t_o) be a SCSC solution of the problems (1) and (2), for a selected DMU_o.*

Then, DMU_o belongs to

E iff $\theta_o = 1, S_o = 0$, and $t_{oj} > 0$ for all $j \neq o$.

E' iff $\theta_o = 1, S_o = 0$, and $t_{oj} = 0$ for some $j \neq o$.

F iff $\theta_o = 1, S_o \neq 0$.

NE iff $\theta_o < 1, S_o = 0$, and for some DMU_k $\lambda_{ok} \neq 0$ and $\lambda_{oj} = 0$ for all $j \neq k$.

NE' iff $\theta_o < 1, S_o = 0$, and more than one λ_{oj} is nonzero.

NF iff $\theta_o < 1, S_o \neq 0$.

The problems considered come from Thompson, Langeimeier, C. Lee, E. Lee and Thrall [17] and Thompson, Lee and Thrall [18].

We present a summary of the results obtained for the following problems:

(P1) Kansas Farming Problem: Problem I from [17]. This problem consists of 32 wheat farms (32 $DMU's$), 4 inputs and 1 output, with normalized output.

(P2) Oil-Gas Firms Problem: Problem II (Ratio case) from [18]. This problem consists of 30 Oil-Gas Extraction firms (30 $DMU's$), 4 inputs and 2 outputs.

Following the notation used in the paper, we denote by (θ, λ, S) and (u, v, t) solutions for the linear programming problems (1) and (2) respectively.

As an illustration of the output obtained by using the algorithm, we show the results for DMU_1, DMU_8 and DMU_{32} from problem P1, in Tables 1, 2 and 3. By counting the number of nonzero values it may be observed that the solutions obtained are SCSC solutions in each problem, so any of these Tables may be directly used to classify the respective $DMU's$ using Lemma 6.2.

Tables 4, 7 contain the summarized information needed for Lemma 6.2 in order to classify the different $DMU's$ for problems P1 and P2 respectively.

Tables 5, 8 contain the dual or multiplier solutions for the $DMU's$ in E, for problems P1 and P2 respectively.

Tables 6, 9 contain the nonzero values of the primal or envelopment solutions for all the $DMU's$ for problems P1 and P2 respectively.

Columns 2 and 3 of Tables 4, 7 show that the solutions obtained for each of the $DMU's$ are SCSC solutions.

For the Kansas Farming Problem (problem P1) we obtain:

$$E = \{DMU_j, \text{ for } j = 8, 9, 14, 15, 16, 31\}$$

$$NE' = \{DMU_{32}\}$$

All other $DMU's$ belong to NF.

For the Oil-gas Firms Problem (problem P2) we obtain:

$E = \{DMU_j,\ \text{for } j = 1, 2, 3, 5, 9, 11, 12, 15, 19, 23, 24, 26, 29\}$

All other $DMU's$ belong to NF.

6.2 Sensitivity Analysis

For each extreme efficient DMU_o to find a SCSC solution centrally positioned in the multiplier space $\mathcal{W}_o$ is of extreme importance in evaluating data irregularities (see Thompson, Dharmapala, and Thrall [16] for a detailed discussion) .

The Long-Step Shrinking-Neighborhood Algorithm presented in this work was designed for computing a SCSC solution being a center-most solution in the sense defined in section 3. This is the first time that a centrality condition is considered for solving DEA linear programming problems.

Thompson, Dharmapala, Diaz, and Thrall [15] studied the effect of using the solutions obtained by the Long-Step Shrinking-Neighborhood Algorithm to analyze the data. They made a multiplier sensitivity analysis to the data for the Oil-Gas Firms Problem (Problem (P2)) considering the analytic center solutions from Table 7 and the SCSC solutions calculated using the simplex algorithm repeatedly (simplex derived centers, see [4]). They observed that the solutions obtained by using the Long-Step Shrinking-Neighborhood Algorithm improved the stability of the sensitivity results. This study suggests that the centrality condition satisfied by the analytic center solutions can be used to contribute additional robustness in the DEA theory.

Table 1: RESULTS FOR THE KANSAS FARM PROBLEM: DMU1

S	θ	λ	w	$v^T x_o - 1$	t
0.477352519e-11	0.634866211e+00	0.829983403e-11	0.634866211e+00	0.477352519e-11	0.365133789e+00
0.731225593e-11		0.829983403e-11	0.414448000e+00		0.365133789e+00
0.527656801e-03		0.133743123e-11	0.574341096e-08		0.226594817e+01
0.304820975e-10		0.502491263e-10	0.994206469e-01		0.603104984e-01
0.119477025e-11		0.469009492e-11	0.253651265e+01		0.646159598e+00
		0.849414685e-12			0.356780958e+01
		0.764230363e-11			0.396549260e+00
		0.789475664e-11			0.383868685e+00
		0.113349502e+00			0.267363314e-10
		0.419171978e-10			0.722984839e-01
		0.441933086e-11			0.685748576e+00
		0.579963577e-11			0.522541409e+00
		0.381167080e-11			0.795071246e+00
		0.926337792e-11			0.327153861e+00
		0.257668182e-10			0.117614438e+00
		0.110093286e+00			0.275271086e-10
		0.776557212e+00			0.390254550e-11
		0.149825333e-11			0.202272191e+01
		0.500714380e-11			0.605245220e+00
		0.438855748e-11			0.690557174e+00
		0.190747980e-11			0.158877166e+01
		0.318297036e-11			0.952113750e+00
		0.191582653e-11			0.158184982e+01
		0.274818353e-11			0.110274653e+01
		0.417850005e-11			0.725272183e+00
		0.196422959e-10			0.154286946e+00
		0.927996340e-12			0.326569160e+01
		0.425721304e-11			0.711862390e+00
		0.515330932e-11			0.588078390e+00
		0.526099609e-11			0.576041077e+00
		0.127342959e-11			0.237983308e+01
		0.552632292e-11			0.548384504e+00
		0.821066350e-11			0.369099263e+00

Table 2: RESULTS FOR THE KANSAS FARM PROBLEM: DMU8

S	θ	λ	w	$v^T x_o - 1$	t
0.570105636D-13	0.100000000D+01	0.440773806D-13	0.100000000D+01	0.570105636D-13	0.129341995D+01
0.897971989D-13		0.753690564D-14	0.634881314D+00		0.756418700D+01
0.459608832D-13		0.869418715D-13	0.124041489D+01		0.655731958D+00
0.132356100D-12		0.299216371D-13	0.430736201D+00		0.190532902D+01
0.246057010D-12		0.896999832D-14	0.231696564D+00		0.635569391D+01
		0.330838528D-13			0.172321416D+01
		0.653123867D-13			0.872890527D+00
		0.100000000D+01			0.570105636D-13
		0.853612776D-13			0.667873833D+00
		0.303352400D-13			0.187935100D+01
		0.446189993D-13			0.127771946D+01
		0.343073471D-13			0.166175961D+01
		0.337956822D-13			0.168691856D+01
		0.706490508D-13			0.806954417D+00
		0.459792663D-12			0.123991895D+00
		0.100228656D-12			0.568805030D+00
		0.940300020D-14			0.606301845D+01
		0.536671569D-13			0.106229894D+01
		0.624274326D-13			0.913229349D+00
		0.133021584D-13			0.428581301D+01
		0.341685228D-13			0.166851122D+01
		0.111535134D-13			0.511144440D+01
		0.319042607D-13			0.178692633D+01
		0.225900411D-13			0.252370341D+01
		0.798952502D-13			0.713566370D+00
		0.978546423D-14			0.582604589D+01
		0.415468840D-13			0.137219830D+01
		0.267488973D-13			0.213132388D+01
		0.281631954D-13			0.202429316D+01
		0.684349345D-14			0.833062296D+01
		0.205195777D-13			0.277834976D+01
		0.449115117D-13			0.126939757D+01

Table 3: RESULTS FOR THE KANSAS FARM PROBLEM: DMU32

S	θ	λ	w	$v^T x_o - 1$	t
0.285312080D-14	0.647165747D+00	0.488299879D-14	0.647165747D+00	0.285312080D-14	0.378136907D+00
0.734008859D-14		0.110589947D-14	0.251555827D+00		0.166962921D+01
0.258825573D-15		0.156970572D-13	0.713392435D+01		0.117629823D+00
0.273494886D-13		0.294661638D-14	0.675128551D-01		0.626631302D+00
0.125575227D-14		0.486222149D-15	0.147038720D+01		0.379752765D+01
		0.309173120D-14			0.597219465D+00
		0.696961199D-14			0.264927525D+00
		0.451224452D-01			0.409207003D-13
		0.158130335D-13			0.116767099D+00
		0.310002443D-14			0.595621777D+00
		0.421937525D-14			0.437610297D+00
		0.340843431D-14			0.541727341D+00
		0.793726566D-14			0.232629489D+00
		0.126394816D+00			0.146085268D-13
		0.235073169D+00			0.785475462D-14
		0.593409570D+00			0.311158119D-14
		0.127445765D-14			0.144880613D+01
		0.384180336D-14			0.480618575D+00
		0.362425758D-14			0.509467667D+00
		0.159914870D-14			0.115464062D+01
		0.237814921D-14			0.776419767D+00
		0.128104662D-14			0.144135430D+01
		0.166311639D-14			0.111023021D+01
		0.310344576D-14			0.594965145D+00
		0.808334773D-14			0.228425415D+00
		0.701951308D-15			0.263044179D+01
		0.348980844D-14			0.529095534D+00
		0.728381201D-15			0.253499411D+01
		0.266476701D-14			0.692909381D+00
		0.108433758D-14			0.170282953D+01
		0.478037926D-14			0.386254302D+00
		0.523317122D-14			0.352834253D+00

Table 4: RESULTS FOR THE KANSAS FARM PROBLEM

DMU	θ	$\min_j \lvert s_j + w_j \rvert$	$min_j \lvert t_j + \lambda_j \rvert$	$\lVert S \rVert_1$	$\{j : \lambda_j > 0\}$	CLASS
1	6.3487e-01	5.2766e-04	6.0310e-02	5.2766e-04	8 15 16	NF
2	4.5162e-01	2.3815e-02	2.1789e-02	5.4174e+00	8 14	NF
3	9.6969e-01	4.3558e-03	3.0309e-02	4.3558e-03	9 15 16	NF
4	5.3024e-01	3.4651e-01	2.3649e-02	4.1843e-01	8 14 16	NF
5	1.5106e-01	3.4519e-03	8.4651e-03	3.4519e-03	8 15 16	NF
6	7.8899e-01	2.9878e-02	4.7398e-02	1.8294e+00	8 16	NF
7	7.7507e-01	3.3121e-02	3.7395e-02	1.9713e-01	8 14	NF
8	1.0000e+00	2.3170e-01	1.2399e-01	5.7118e-13	8	E
9	1.0000e+00	2.6241e-02	7.9946e-02	4.6025e-12	9	E
10	8.1561e-01	1.1183e-02	1.7065e-01	3.1344e+00	8	NF
11	6.3743e-01	7.6213e-03	3.0754e-02	4.9780e-01	8 14	NF
12	6.8896e-01	2.7369e-02	5.0417e-02	5.5087e-01	14 15	NF
13	9.4779e-01	6.1937e-01	1.6970e-02	2.7653e+00	8 14 16	NF
14	1.0000e+00	1.7030e-02	2.0229e-01	4.4938e-12	14	E
15	1.0000e+00	2.8111e-02	6.0560e-01	1.2408e-12	15	E
16	1.0000e+00	2.9851e-02	7.2386e-02	4.7218e-12	16	E
17	4.2907e-01	2.2462e-02	1.8105e-01	2.7632e+00	14	NF
18	6.4311e-01	3.9577e-02	4.7062e-02	8.5941e-02	14 15	NF
19	6.8578e-01	5.1246e-02	5.0184e-02	1.4179e-01	14 15	NF
20	4.3438e-01	4.0239e-02	8.9820e-03	1.1709e+00	8 14	NF
21	5.0561e-01	4.9383e-02	7.3112e-02	4.9383e-02	8 14 15	NF
22	3.6053e-01	2.3561e-01	1.9861e-02	1.5674e+00	8 14 16	NF
23	3.7680e-01	4.9142e-04	2.6901e-01	8.3244e-03	8 15	NF
24	6.0339e-01	3.9431e-01	3.3238e-02	1.5156e+00	8 14 16	NF
25	8.0449e-01	8.0235e-03	5.4330e-02	8.0235e-03	8 15 16	NF
26	2.3553e-01	3.6162e-02	1.7236e-02	1.6238e-01	14 15	NF
27	6.8683e-01	1.1417e-02	5.0261e-02	5.8361e-01	14 15	NF
28	7.4989e-01	3.9878e-02	3.6180e-02	7.8296e-01	8 14	NF
29	5.5820e-01	1.0230e-02	3.3534e-02	4.2322e-01	8 16	NF
30	4.5968e-01	2.1411e-02	2.2178e-02	5.9699e+00	8 14	NF
31	1.0000e+00	9.4731e-05	9.5463e-04	1.8868e-11	31	E
32	6.4717e-01	6.7513e-02	4.5122e-02	3.9057e-14	8 14 15 16	NE'

Table 5: RESULTS FOR THE KANSAS FARM PROBLEM

DUAL MULTIPLIERS $w = (u, v)$ FOR THE DMU's IN E					
DMU	U1	V1	V2	V3	V4
8	1.0000e+00	6.3488e-01	1.2404e+00	4.3074e-01	2.3170e-01
9	1.0000e+00	3.4861e-02	5.9672e-01	2.6241e-02	2.8584e+01
14	1.0000e+00	3.9452e-02	2.0995e+01	1.7030e-02	1.1569e+01
15	1.0000e+00	2.8111e-02	1.0096e+00	1.0009e+00	2.5996e+00
16	1.0000e+00	4.8521e-01	2.2580e+00	2.9851e-02	1.4103e+01
31	1.0000e+00	1.9060e-01	4.1892e+00	9.4731e-05	2.0512e+01

Table 6: RESULTS FOR THE KANSAS FARM PROBLEM

DMU	NONZERO PRIMAL VARIABLES λ			
1	$\lambda_8 = 1.1335e-01$	$\lambda_{15} = 1.1009e-01$	$\lambda_{16} = 7.7656e-01$	-
2	$\lambda_8 = 3.3197e-01$	$\lambda_{14} = 6.6803e-01$	-	-
3	$\lambda_8 = 7.3275e-01$	$\lambda_{14} = 1.7646e-01$	$\lambda_{16} = 9.0795e-02$	-
4	$\lambda_8 = 3.7650e-01$	$\lambda_{14} = 2.3649e-02$	$\lambda_{16} = 5.9985e-01$	-
5	$\lambda_8 = 1.6781e-01$	$\lambda_{14} = 8.2373e-01$	$\lambda_{16} = 8.4651e-03$	-
6	$\lambda_8 = 4.0857e-01$	$\lambda_{16} = 5.9143e-01$	-	-
7	$\lambda_8 = 5.3739e-01$	$\lambda_{14} = 4.6261e-01$	-	-
8	$\lambda_8 = 1.0000e+00$	-	-	-
9	$\lambda_9 = 1.0000e+00$	-	-	-
10	$\lambda_8 = 1.0000e+00$	-	-	-
11	$\lambda_8 = 7.0385e-01$	$\lambda_{14} = 2.9615e-01$	-	-
12	$\lambda_{14} = 5.4344e-01$	$\lambda_{15} = 4.5656e-01$	-	-
13	$\lambda_8 = 3.8210e-01$	$\lambda_{14} = 1.6970e-02$	$\lambda_{16} = 6.0093e-01$	-
14	$\lambda_{14} = 1.0000e+00$	-	-	-
15	$\lambda_{15} = 1.0000e+00$	-	-	-
16	$\lambda_{16} = 1.0000e+00$	-	-	-
17	$\lambda_{14} = 1.0000e+00$	-	-	-
18	$\lambda_{14} = 2.4369e-01$	$\lambda_{15} = 7.5631e-01$	-	-
19	$\lambda_{14} = 1.9701e-01$	$\lambda_{15} = 8.0299e-01$	-	-
20	$\lambda_1 = 4.3438e-01$	$\lambda_9 = 8.9820e-03$	$\lambda_{15} = 9.9102e-01$	-
21	$\lambda_8 = 2.5511e-01$	$\lambda_{14} = 3.5935e-01$	$\lambda_{15} = 3.8554e-01$	-
22	$\lambda_8 = 1.0156e-01$	$\lambda_{14} = 7.2329e-02$	$\lambda_{16} = 8.2611e-01$	-
23	$\lambda_8 = 6.1535e-01$	$\lambda_{15} = 3.8465e-01$	-	-
24	$\lambda_8 = 2.9293e-01$	$\lambda_{14} = 2.9882e-01$	$\lambda_{16} = 4.0825e-01$	-
25	$\lambda_8 = 5.4330e-02$	$\lambda_{15} = 3.8062e-01$	$\lambda_{16} = 5.6505e-01$	-
26	$\lambda_{14} = 5.9808e-01$	$\lambda_{15} = 4.0192e-01$	-	-
27	$\lambda_{14} = 2.0409e-01$	$\lambda_{15} = 7.9591e-01$	-	-
28	$\lambda_8 = 8.8701e-02$	$\lambda_{14} = 9.1130e-01$	-	-
29	$\lambda_8 = 1.2523e-01$	$\lambda_{16} = 8.7477e-01$	-	-
30	$\lambda_8 = 1.0537e-01$	$\lambda_{14} = 8.9463e-01$	-	-
31	$\lambda_{31} = 1.0000e+00$	-	-	-
32	$\lambda_8 = 4.5122e-02$	$\lambda_{14} = 1.2639e-01$	$\lambda_{15} = 2.3507e-01$	$\lambda_{16} = 5.9341e-01$

Table 7: RESULTS FOR THE OIL/GAS FIRMS PROBLEM

DMU	θ	$\min_j \lvert s_j + w_j \rvert$	$min_j \lvert t_j + \lambda_j \rvert$	$\lVert S \rVert_1$	$\{j : \lambda_j > 0\}$	CLASS
1	1.0000e+00	1.1395e-04	1.4141e-02	7.5891e-11	1	E
2	1.0000e+00	2.1788e-03	4.5875e-02	5.4520e-13	2	E
3	1.0000e+00	2.5435e-03	3.3921e-03	4.6569e-11	3	E
4	7.2614e-01	1.1798e-01	7.8232e-03	1.5079e+00	9 23 26	NF
5	1.0000e+00	3.0406e-03	2.3723e-01	2.3705e-14	5	E
6	9.4024e-01	1.4180e-03	3.9281e-03	5.9584e+00	9 19 26	NF
7	7.9612e-01	4.1168e-04	1.2080e-02	3.7405e+00	15 19 26 29	NF
8	5.5799e-01	7.8339e-04	4.3760e-02	4.4637e-01	11 19 23 26	NF
9	1.0000e+00	1.1820e-03	1.5607e-02	1.2403e-11	9	E
10	6.4746e-01	6.5483e-03	2.5451e-03	7.1769e-02	1 5 24 26	NF
11	1.0000e+00	1.7149e-04	4.4009e-04	8.2657e-18	11	E
12	1.0000e+00	2.4588e-03	2.0083e-02	7.4641e-15	12	E
13	6.0141e-01	6.7137e-04	1.0454e-02	1.2281e+00	15 19 26 29	NF
14	2.6143e-01	8.7400e-03	2.7892e-03	1.0903e+00	1 26	NF
15	1.0000e+00	1.6835e-03	2.5313e-01	5.7282e-14	15	E
16	3.9684e-01	8.6391e-03	4.2020e-03	9.4331e-01	11 19 23 26	NF
17	7.9127e-01	3.2060e-03	6.6778e-03	5.3531e-01	9 19 26	NF
18	6.1208e-01	1.5648e-02	4.7808e-03	1.5648e-02	2 19 23 26	NF
19	1.0000e+00	4.4342e-02	4.6668e-01	6.1287e-15	19	E
20	5.4550e-01	2.3840e-02	1.5682e-02	7.4206e-02	2 19 23 26	NF
21	5.7068e-01	9.0396e-03	6.3437e-03	4.3583e-01	9 26	NF
22	9.1175e-01	1.3689e-02	1.1081e-02	3.0801e+00	15 26 29	NF
23	1.0000e+00	2.7210e-02	1.0000e+00	1.0015e-15	23	E
24	1.0000e+00	2.6999e-05	5.3297e-03	4.0545e-14	24	E
25	8.5307e-01	1.6499e-03	4.9890e-03	3.3016e+00	2 19 23 26	NF
26	1.0000e+00	1.6248e-03	3.2859e-02	1.7429e-14	26	E
27	4.6902e-01	1.8874e-02	1.1944e-02	1.8138e+00	15 26 29	NF
28	5.3820e-01	3.8300e-03	8.3330e-03	9.1593e-02	11 19 23 26	NF
29	1.0000e+00	1.1265e-03	2.6538e-02	1.4605e-14	29	E
30	6.7486e-01	4.4651e-03	4.4619e-03	4.7914e-01	15 26 29	NF

Table 8: RESULTS FOR THE OIL/GAS FIRMS PROBLEM

DUAL MULTIPLIERS $w = (u, v)$ FOR THE DMU's IN E						
DMU	U1	U2	V1	V2	V3	V4
1	1.0053e-01	7.5247e-02	1.1407e-02	8.4621e-02	1.1395e-04	5.8077e-02
2	5.2761e-01	3.5953e-01	1.6787e-02	1.3186e+00	2.1788e-03	7.0068e-02
3	2.1734e+00	3.7393e-02	3.3891e-03	2.9802e-01	3.3934e-02	2.5435e-03
4	1.2225e+00	2.6358e-01	4.7415e-03	3.0406e-03	3.7762e-03	8.3022e-01
5	9.8530e+00	7.1545e-02	2.0850e-02	1.9476e+00	7.9175e-02	1.1820e-03
6	7.6593e+00	1.7115e+00	6.4374e-01	1.1002e+00	2.4027e-01	1.7149e-04
7	2.6326e+00	2.4588e-03	2.5770e-03	7.0159e-03	3.8157e-02	2.7267e-01
8	3.8882e+00	3.1194e-01	7.4720e-03	2.7055e-02	9.6971e-01	1.6835e-03
9	7.7356e+00	2.5336e+00	4.4342e-02	3.2403e+00	1.5156e-01	1.0898e+00
10	6.9897e+01	8.9983e+00	3.7751e-01	2.8099e+02	5.8142e-02	2.7210e-02
11	3.1337e-01	5.5737e-04	3.5389e-03	4.1623e-04	2.6999e-05	7.3642e-02
12	7.6442e-01	2.8138e-02	1.7083e-01	1.0578e-02	1.6248e-03	3.4998e-02
13	1.1507e+01	2.5467e-01	4.5919e-02	1.1265e-03	1.1704e+00	9.3267e-02

Table 9: RESULTS FOR THE OIL/GAS FIRMS PROBLEM

DMU	NONZERO PRIMAL VARIABLES λ			
1	$\lambda_1 = 1.0000e+00$	-	-	-
2	$\lambda_2 = 1.0000e+00$	-	-	-
3	$\lambda_3 = 1.0000e+00$	-	-	-
4	$\lambda_9 = 7.8232e-03$	$\lambda_{23} = 2.6942e+00$	$\lambda_{26} = 2.6125e-02$	-
5	$\lambda_5 = 1.0000e+00$	-	-	-
6	$\lambda_9 = 5.4611e-01$	$\lambda_{19} = 4.8105e-01$	$\lambda_{26} = 1.1783e-01$	-
7	$\lambda_{15} = 8.9817e-02$	$\lambda_{19} = 1.1871e+00$	$\lambda_{26} = 1.4027e-01$	$\lambda_{29} = 2.5732e-02$
8	$\lambda_{11} = 1.6993e-01$	$\lambda_{19} = 1.0968e+00$	$\lambda_{23} = 8.3803e-01$	$\lambda_{26} = 4.7418e-02$
9	$\lambda_9 = 1.0000e+00$	-	-	-
10	$\lambda_1 = 2.5451e-03$	$\lambda_5 = 1.1024e-01$	$\lambda_{24} = 5.1342e-03$	$\lambda_{26} = 2.4428e-02$
11	$\lambda_{11} = 1.0000e+00$	-	-	-
12	$\lambda_{12} = 1.0000e+00$	-	-	-
13	$\lambda_{15} = 5.6227e-02$	$\lambda_{19} = 7.9042e-01$	$\lambda_{26} = 5.1826e-02$	$\lambda_{29} = 1.0454e-02$
14	$\lambda_1 = 2.7892e-03$	$\lambda_{26} = 1.8558e-02$	-	-
15	$\lambda_1 = 1.0000e+00$	-	-	-
16	$\lambda_{11} = 1.9063e-02$	$\lambda_{19} = 4.6213e-02$	$\lambda_{23} = 3.0811e-02$	$\lambda_{26} = 4.2020e-03$
17	$\lambda_9 = 9.4548e-03$	$\lambda_{19} = 1.8473e-01$	$\lambda_{26} = 6.6778e-03$	-
18	$\lambda_2 = 4.7808e-03$	$\lambda_{19} = 8.0622e-02$	$\lambda_{23} = 2.0079e-01$	$\lambda_{26} = 9.2983e-03$
19	$\lambda_{19} = 1.0000e+00$	-	-	-
20	$\lambda_2 = 1.2193e-01$	$\lambda_{19} = 1.7270e-01$	$\lambda_{23} = 5.1396e-01$	$\lambda_{26} = 1.5682e-02$
21	$\lambda_9 = 1.4964e-01$	$\lambda_{26} = 6.3437e-03$	-	-
22	$\lambda_{15} = 4.0172e-02$	$\lambda_{26} = 1.1081e-02$	$\lambda_{29} = 1.7340e-0223$	$\lambda_{23} = 1.0000e+00$
24	$\lambda_{24} = 1.0000e+00$	-	-	-
25	$\lambda_2 = 1.3020e+00$	$\lambda_{19} = 9.4998e+00$	$\lambda_{23} = 2.3073e+01$	$\lambda_{26} = 5.5866e-01$
26	$\lambda_{26} = 1.0000e+00$	-	-	-
27	$\lambda_{15} = 2.3529e-02$	$\lambda_{26} = 1.1944e-02$	$\lambda_{29} = 4.8987e-02$	-
28	$\lambda_{11} = 3.6102e-02$	$\lambda_{19} = 6.6127e-02$	$\lambda_{23} = 7.3328e-01$	$\lambda_{26} = 8.3330e-03$
29	$\lambda_{29} = 1.0000e+00$	-	-	-
30	$\lambda_{15} = 4.4931e-02$	$\lambda_{26} = 1.0155e-02$	$\lambda_{29} = 4.4619e-03$	-

7 Concluding Remarks

In this paper a primal-dual interior-point algorithm for solving the DEA linear programming problems was proposed. The algorithm is a modification of the Kojima-Mizuno-Yoshise primal-dual algorithm. The sequence of iterates generated by the algorithm converges to the solution set while approaching the central path; hence the computed solution is the analytic center of the solution set. The approach to the central path is done in such a way that long steps can be taken when the gap is not small, but close to the solution set only small steps are allowed because of the use of gradually shrinking neighborhoods of the central path.

Numerical results were presented for some real world DEA problems in order to illustrate the performance of the algorithm. The solution computed by the algorithm can be effectively used for the classification of the $DMU's$ for DEA problems.

The effective use of the Lemma10A from [4] for classifying the $DMU's$ strongly depends on the computation of a SCSC solution being far of the relative boundary of the solution set, in the sense introduced in Section 2. Since the solution computed by the algorithm proposed in the present work is a center-most solution in the sense that it maximizes the product of the positive components among all the SCSC solutions, it seems to be a good choice for the DEA purposes, although it may not be the best solution choice. To our knowledge, this is the first time that a centrality requirement is considered for finding solutions of the DEA linear programming problems. Further research is needed to understand the relationship between the solution here obtained and the DEA-center solution introduced in Section 2, as well as the relationship between the optimal value d_o^c and $d_o^* = d_o(w*)$ with $w*$ the dual multiplier of the analytic center solution and d_o the sensitivity function also introduced in Section 2.

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